

# WEINGARTEN TYPE SURFACES IN $\mathbb{H}^2 \times \mathbb{R}$ AND $\mathbb{S}^2 \times \mathbb{R}$

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**ABSTRACT.** In this article, we study complete surfaces  $\Sigma$ , isometrically immersed in the product space  $\mathbb{H}^2 \times \mathbb{R}$  or  $\mathbb{S}^2 \times \mathbb{R}$  having positive extrinsic curvature  $K_e$ . Let  $K_i$  denote the intrinsic curvature of  $\Sigma$ . Assume that the equation  $aK_i + bK_e = c$  holds for some real constants  $a \neq 0$ ,  $b > 0$  and  $c$ . The main result of this article state that when such a surface is a topological sphere it is rotational.

## 1. INTRODUCTION

The Hopf's holomorphic quadratic differential form, defined on surface having constant mean curvature in  $\mathbb{R}^3$ , enables Hopf to give a proof that topological spheres in  $\mathbb{R}^3$  having constant mean curvature are rotational. A few years ago, Abresch and Rosenberg (see [1], [2]) discovered a holomorphic quadratic differential form on constant mean curvature surfaces in the homogeneous 3-manifolds. With the aid of this quadratic form, they extended the Hopf's result to constant mean curvature topological spheres immersed in such homogeneous spaces.

In the product spaces  $\mathbb{H}^2 \times \mathbb{R}$  and  $\mathbb{S}^2 \times \mathbb{R}$ , Aledo, Espinar and Gálvez [3], associated a holomorphic quadratic differential form to constant intrinsic curvature (Gaussian curvature) surfaces immersed in such product spaces, which enabled them to extend the classical Liebmann Theorem, that in the euclidean space  $\mathbb{R}^3$  ensure that, the round spheres are the unique complete surfaces of positive constant intrinsic curvature. For complete surfaces immersed in  $\mathbb{H}^2 \times \mathbb{R}$  or  $\mathbb{S}^2 \times \mathbb{R}$  having positive extrinsic curvature, Gálvez, Espinar and Rosenberg [8], proved that such surfaces are embedded and homeomorphic to either the euclidean sphere  $\mathbb{S}^2$  or to the euclidean plane  $\mathbb{R}^2$ . Moreover, they construct a quadratic differential form on positive constant extrinsic surfaces which vanishes identically or its zeros are isolated with negative index. As a consequence, they proved that the complete immersions having positive constant extrinsic curvature in the product spaces  $\mathbb{H}^2 \times \mathbb{R}$  and  $\mathbb{S}^2 \times \mathbb{R}$  are rotational sphere.

In this article, we consider complete surfaces  $\Sigma$ , isometrically immersed in the product spaces  $\mathbb{H}^2 \times \mathbb{R}$  or  $\mathbb{S}^2 \times \mathbb{R}$  having positive extrinsic curvature (non-constant) such that,

$$aK_i + bK_e = c,$$

where  $K_i$ ,  $K_e$  are the intrinsic and the extrinsic curvatures, respectively, and  $a \neq 0$ ,  $b > 0$  and  $c$  are real constants. Our goal is to prove that if  $\Sigma$  is a topological sphere, then  $\Sigma$  is rotational. In order to obtain this result, we first construct a quadratic differential form  $Qdz^2$  which vanishes identically or its zeros are isolated with negative index, this quadratic form exists if  $a + b \neq 0$ ,  $2a + b \neq 0$ , see subsection 4.2. Moreover, we obtain vertical and horizontal height estimates which enable us to realize when  $\Sigma$  is a topological sphere, see Section 4 and Section 5. In Section 6, we prove the main theorem.

The article is organized as follows: In section 2, we give the definition of Weingarten type surfaces. Section 3 is devoted to the study of rotational examples. In section 4, we construct a quadratic differential form on a Weingarten type surface which vanishes identically or its zeros are isolated

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with negative index. Also, we establish horizontal and vertical height estimates. In Section 5 we study the non-existence of properly embedded surfaces having finite topology and one top end. In section 6, we prove the main theorem of this article.

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## 2. WEINGARTEN TYPE SURFACES HAVING POSITIVE EXTRINSIC CURVATURE

For  $\epsilon \in \{-1, 1\}$ , we denote by  $M^2(\epsilon)$  the complete, connected, simply-connected, 2-dimensional space form having sectional curvature  $\epsilon$ . That is, for  $\epsilon = 1$ ,  $M^2(\epsilon)$  label the canonical euclidean unit sphere  $S^2$  and for  $\epsilon = -1$ ,  $M^2(\epsilon)$  denotes the complete, connected, simply-connected hyperbolic plane  $\mathbb{H}^2$  having sectional curvature  $-1$ . Also, we denote by  $M^2(\epsilon) \times \mathbb{R}$  the product space (where  $\mathbb{R}$  is the real line), endowed with the product metric.

Recall that, the surface  $\Sigma$  is called a Weingarten surface if its two principal curvatures  $k_1$  and  $k_2$  are not independent one of another or, equivalently, if there exist a relation of the form  $W(k_1, k_2) = 0$  for a smooth real function  $W : \mathcal{D} \subset \mathbb{R}^2 \rightarrow \mathbb{R}$  defined on a set  $\mathcal{D}$ .

In this article, we study complete, connected, surfaces  $\Sigma$  isometrically immersed in the product space  $M^2(\epsilon) \times \mathbb{R}$  whose intrinsic and extrinsic curvature are linearly related, more precisely,

**Definition 2.1.** Let  $\varphi : \Sigma \rightarrow M^2(\epsilon) \times \mathbb{R}$  be an isometric immersion from a connected surface having intrinsic curvature  $K_i$  and extrinsic curvature  $K_e$ . We say that  $\Sigma$  is a Weingarten type surface or simply a  $W$ -surface if there exist three real numbers,  $a \neq 0$ ,  $b > 0$  and  $c$  such that,

$$(2.1) \quad aK_i + bK_e - c = 0.$$

*Remark 2.2.* The assumption  $b > 0$  is not a restriction since we can multiply by  $-1$  the equation 2.1 if necessary.

For simplicity, we treat properties of an immersion  $\varphi$  as those of  $\Sigma$  and denote merely by  $\Sigma$  the image  $\varphi(\Sigma)$ . For example, we call  $\Sigma$  a  $W$ -surface in  $M^2(\epsilon) \times \mathbb{R}$  instead of saying that the immersion  $\varphi$  is a  $W$ -surface in  $M^2(\epsilon) \times \mathbb{R}$ .

Let  $\varphi : \Sigma \rightarrow M^2(\epsilon) \times \mathbb{R}$  be an isometric immersion from an orientable surface  $\Sigma$  into the product space  $M^2(\epsilon) \times \mathbb{R}$ . We chose a global unit normal vector field  $N$  and as usual, we denote by  $\nu = \langle N, \frac{\partial}{\partial t} \rangle$  the angle function of  $\Sigma$ , here  $\frac{\partial}{\partial t}$  denotes the tangent vector field to the real line  $\mathbb{R}$ . From [5], we have that the Gauss equation for such an immersed surface into the product space  $M^2(\epsilon) \times \mathbb{R}$  is given by

$$(2.2) \quad K_i = K_e + \epsilon \nu^2.$$

As a consequence of the Gauss equation, we have

**Lemma 2.3.** Let  $\varphi : \Sigma \rightarrow M^2(\epsilon) \times \mathbb{R}$  be an isometric immersion. Assume that  $\Sigma$  is a complete  $W$ -surface having positive extrinsic curvature  $K_e$ . Then:

- (1) Suppose  $a + b > 0$ , then
  - i) If  $c \leq 0$ ,  $\Sigma$  is not compact.
  - ii) If  $\epsilon = -1$  and  $c > b$ ,  $\Sigma$  is closed.
  - iii) If  $\epsilon = 1$  and  $c > 0$ ,  $\Sigma$  is closed.
- (2) Suppose  $a + b < 0$ , then
  - i) If  $c \geq 0$ ,  $\Sigma$  is not compact.
  - ii) If  $\epsilon = -1$  and  $c < 0$ ,  $\Sigma$  is closed.
  - iii) If  $\epsilon = 1$  and  $c < -b$ ,  $\Sigma$  is closed.

- iv) If  $\epsilon = 1$  and  $-b \leq c < 0$ ,  $\Sigma$  cannot be closed.  
 (3) For  $a + b = 0$ , the angle function is constant.

*Proof.* As the extrinsic curvature of the  $W$ -surface is positive,  $\Sigma$  is orientable and we choose the unit global normal vector field such that the second fundamental form is definite positive.

For  $\mathbb{H}^2 \times \mathbb{R}$ , if  $a + b < 0$ , it is clear that for  $c < 0$ , the intrinsic curvature satisfies  $K_i \geq \frac{c}{a+b} > 0$ . Then, from the Bonnet-Myers theorem,  $\Sigma$  must be compact. On the other hand, for  $a + b < 0$  and  $c \geq 0$ , if  $\Sigma$  were compact, there would exist a point  $p \in \Sigma$  such that  $\nu(p) = 0$ , it would imply that the extrinsic curvature satisfies  $K_e(p) \leq 0$ , which contradicts our assumption. The proof of the other cases are similar.

From equation (2.1) and (2.2), we conclude that  $a + b = 0$  implies that the angle function is constant.  $\square$

*Remark 2.4.* Since surfaces having constant angle were treated in [6] and [7], from now on, we omit this case.

### 3. COMPLETE ROTATIONAL SURFACES OF WEINGARTEN TYPE IN $M^2(\epsilon) \times \mathbb{R}$

In this section, we deal with complete  $W$ -surfaces having positive extrinsic curvature, which are invariant by one-parameter group of rotations of the ambient space  $M^2(\epsilon) \times \mathbb{R}$ .

For  $\epsilon \in \{-1, 1\}$ , let us consider the 4-dimensional space  $\mathbb{R}_\epsilon^3 \times \mathbb{R}$ , endowed with the metric  $ds^2 = \epsilon dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$ . And, let us identify the product space  $M^2(\epsilon) \times \mathbb{R}$  as being the sub-manifold of  $\mathbb{R}_\epsilon^3 \times \mathbb{R}$ , given by

$$M^2(\epsilon) \times \mathbb{R} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}_\epsilon^3 \times \mathbb{R} : \epsilon x_1^2 + x_2^2 + x_3^2 = \epsilon, \text{ and if } \epsilon = -1, x_1 > 0\}.$$

The rotation in  $M^2(\epsilon) \times \mathbb{R}$  is a subgroup of the isometry group of  $M^2(\epsilon) \times \mathbb{R}$  which preserves the orientation and fixes an axis  $\{p\} \times \mathbb{R}$ , with  $p \in M^2(\epsilon) \times \{0\}$ . This subgroup can be identify with the special orthogonal group  $SO(2)$ . Up to isometries, we can assume that the axis is given by  $\{(1, 0, 0)\} \times \mathbb{R}$ .

We consider the plane  $\Pi = \{(x_1, x_2, 0, x_4) \in M^2(\epsilon) \times \mathbb{R}, x_2 \geq 0\}$  and the curve

$$\alpha_\epsilon(u) = \begin{cases} (\cosh k(u), \sinh k(u), 0, h(u)) & \subset \Pi, \text{ if } \epsilon = -1, \\ (\cos k(u), \sin k(u), 0, h(u)) & \subset \Pi, \text{ if } \epsilon = 1. \end{cases}$$

Where  $k(u) \geq 0$  and  $u$  is the arclength of  $\alpha$ , that is,  $(k'(u))^2 + (h'(u))^2 = 1$ . Here  $k'(u)$  denotes the derivative with respect to the variable  $u$ .

In order to obtain a rotational surface, we apply one-parameter group of rotational isometries to the curve  $\alpha_\epsilon$ . Denoting by  $\mathcal{S}$  such a generated surface, we can parametrized  $\mathcal{S}$  by

$$\varphi_\epsilon(u, v) = \begin{cases} (\cosh k(u), \sinh k(u) \cos v, \sinh k(u) \sin v, h(u)), & \text{if } \epsilon = -1, \\ (\cos k(u), \sin k(u) \cos v, \sin k(u) \sin v, h(u)), & \text{if } \epsilon = 1. \end{cases}$$

In order to simplify the expressions, we define the functions

$$\cos_\epsilon k = \begin{cases} \cosh k, & \text{if } \epsilon = -1 \\ \cos k, & \text{if } \epsilon = 1 \end{cases} \quad \text{and} \quad \cot_\epsilon k = \begin{cases} \coth k, & \text{if } \epsilon = -1 \\ \cot k, & \text{if } \epsilon = 1. \end{cases}$$

**3.1. The first integral.** The aim of this section is to classify complete rotational W-surfaces having positive extrinsic curvature. A straightforward computation gives us that the intrinsic and extrinsic curvature function of an isometrically immersed surface which is invariant by rotational isometries in the space  $M^2(\epsilon) \times \mathbb{R}$  are given by

$$\begin{aligned} K_i &= \epsilon(k'(u))^2 - k''(u) \cot_\epsilon k(u), \\ K_e &= -k''(u) \cot_\epsilon k(u). \end{aligned}$$

The Weingarten equation is written as

$$(3.1) \quad (a+b)k''(u) \cot_\epsilon k(u) - \epsilon a(k'(u))^2 = -c,$$

for real numbers  $a \neq 0$ ,  $b > 0$  and  $c$  satisfying  $a+b \neq 0$ . It is direct to check that the first integral of the ordinary differential equation (ODE in short) (3.1) is

$$(3.2) \quad (k'(u))^2 = \epsilon \frac{c}{a} + C_1 (\cos_\epsilon k(u))^{-\frac{2a}{a+b}}$$

for some constant  $C_1$ . Moreover, we can assume that  $\alpha$  cuts the axis orthogonally at  $t = 0$ . Then  $k(0) = 0$  and  $k'(0) = 1$ , in this case, the first integral is given by

$$(3.3) \quad (k'(u))^2 = \epsilon \frac{c}{a} + \frac{a - \epsilon c}{a} (\cos_\epsilon k(u))^{-\frac{2a}{a+b}}.$$

Notice that, the problem of find all complete rotational W-surfaces which cut the axis orthogonally, consist in determine all the admissible expressions of the profile curve  $\alpha_\epsilon$ , we mean, we wish to find all the possible compact (and non-compact) integral curves of the ODE system

$$(3.4) \quad \begin{cases} (k'(u))^2 - \epsilon \frac{c}{a} = \frac{a - \epsilon c}{a} (\cos_\epsilon k(u))^{-\frac{2a}{a+b}}, \\ (k'(u))^2 + (h'(u))^2 = 1. \end{cases}$$

We have the next proposition.

**Proposition 3.1.** *Let  $\mathcal{S}$  be a complete W-surface isometrically immersed into the product space  $M^2(\epsilon) \times \mathbb{R}$  having positive extrinsic curvature  $K_e$  which is invariant by rotational isometries and whose generating curve  $\alpha_\epsilon$  cuts the rotation axis orthogonally. Assume that  $a+b \neq 0$ , then:*

- (1) For  $a+b > 0$ , there are two cases,
  - If  $c > 0$ ,  $\mathcal{S}$  is a rotational topological sphere.
  - If  $c \leq 0$ ,  $\mathcal{S}$  homeomorphic to  $\mathbb{R}^2$ .
- (2) For  $a+b < 0$ , there are two cases,
  - If  $c \geq 0$ ,  $\mathcal{S}$  is homeomorphic to  $\mathbb{R}^2$ .
  - In  $\mathbb{H}^2 \times \mathbb{R}$ , if  $c < 0$ ,  $\mathcal{S}$  is a rotational topological sphere.
  - In  $\mathbb{S}^2 \times \mathbb{R}$ , if  $c < -b$ ,  $\mathcal{S}$  is a rotational topological sphere.
  - In  $\mathbb{S}^2 \times \mathbb{R}$ , if  $-b \leq c < 0$ , there is no rotational surface  $\mathcal{S}$  whose generating curve cuts orthogonally the rotation axis.

*Proof.* It is known that complete surfaces, isometrically immersed in the product spaces  $M^2(\epsilon) \times \mathbb{R}$ , having positive extrinsic curvature are homeomorphic either to a sphere or to the euclidean plane  $\mathbb{R}^2$ , see [8, Theorem 3.1], and [9, Theorem 2.4]. By Lemma 2.3, in order to prove the proposition, we just need to consider two cases. The first one is  $\epsilon = -1$ ,  $a+b > 0$  and  $c > 0$ , and the second is  $\epsilon = 1$ ,  $a+b < 0$  and  $-b \leq c < 0$ .

From (3.4), for  $\epsilon = -1$ ,  $a+b > 0$  and  $c > 0$ , we have

$$(3.5) \quad \left( \frac{dh}{dk} \right)^2 = \left( \frac{a+c}{c} \right) \frac{(\cosh k)^{\frac{2a}{a+b}} - 1}{\left( \sqrt{\frac{a+c}{c}} + (\cosh k)^{\frac{a}{a+b}} \right) \left( \sqrt{\frac{a+c}{c}} - (\cosh k)^{\frac{a}{a+b}} \right)}.$$

That is, we consider the function  $h = h(k)$  as a function of the variable  $k$ ; notice that  $k$  is the hyperbolic distance to the origin in the slice  $\mathbb{H}^2 \times \{0\}$ . This function is defined on the interval  $[0, k_0]$ , where  $k_0$  satisfies

$$(\cosh k_0)^{\frac{a}{a+b}} = \sqrt{\frac{a+c}{c}}.$$

The graph of the function  $h$  has a vertical tangent line at  $k_0$ . In order to obtain a rotational topological sphere, we need to show that the height function  $h(k)$  is bounded and it is of class  $C^2$  at  $k = k_0$ .

Up to isometries of the ambient space, we can assume that  $\frac{dh}{dk} \geq 0$ . We separate the proof in two cases, depending on the sign of  $a$  (recall we are assuming that  $a \neq 0$ ).

i) If  $a > 0$ , then  $\frac{2a}{a+b} > 0$  and  $(\cosh k)^{\frac{2a}{a+b}} - 1 > 0$ . For  $k > 0$ , we set

$$A_1(k) = \sqrt{\left(\frac{a+c}{c}\right) \frac{(\cosh k)^{\frac{2a}{a+b}} - 1}{\left(\sqrt{\frac{a+c}{c}} + (\cosh k)^{\frac{a}{a+b}}\right)}},$$

thus, equation (3.5) implies

$$(3.6) \quad \frac{dh}{dk} = A_1(k) \frac{1}{\sqrt{\sqrt{\frac{a+c}{c}} - (\cosh k)^{\frac{a}{a+b}}}} \frac{2\left(\frac{-a}{a+b}\right)(\cosh k)^{\frac{-b}{a+b}} \sinh k}{2\left(\frac{-a}{a+b}\right)(\cosh k)^{\frac{-b}{a+b}} \sinh k}$$

moreover, if we consider the function

$$A_2(k) = A_1(k) \frac{2}{\left(\frac{a}{a+b}\right)(\cosh k)^{\frac{-b}{a+b}} \sinh k},$$

we can write equation (3.6) as

$$(3.7) \quad \frac{dh}{dk} = A_2(k) \frac{(-1)\left(\frac{-a}{a+b}\right)(\cosh k)^{\frac{-b}{a+b}} \sinh k}{2\sqrt{\sqrt{\frac{a+c}{c}} - (\cosh k)^{\frac{a}{a+b}}}} = A_2(k) \frac{d}{dk} \sqrt{\sqrt{\frac{a+c}{c}} - (\cosh k)^{\frac{a}{a+b}}}.$$

Notice that  $A_1(k)$  and  $A_2(k)$  are bounded functions on the interval  $[0, k_0]$ , then, for each  $0 < \delta < k_0$ , there exist a positive number  $M > 0$ , such that, for all  $k \in [k_0 - \delta, k_0]$ , we have

$$(3.8) \quad \frac{dh}{dk} \leq -M \frac{d}{dk} \sqrt{\sqrt{\frac{a+c}{c}} - (\cosh k)^{\frac{a}{a+b}}}.$$

Integrating (3.8), there exists a constant  $C_1$  large enough, such that

$$(3.9) \quad 0 < h(k) \leq M \left( C_1 - \sqrt{\sqrt{\frac{a+c}{c}} - (\cosh k)^{\frac{a}{a+b}}} \right).$$

The function  $h = h(k)$  is bounded in  $[k_0 - \delta, k_0]$ . From equation (3.5) its graph has a vertical tangent line at  $k = k_0$ , a straightforward computation gives that the function  $h$  is of class  $C^2$  at  $k = k_0$ , that is, its graph has bounded curvature at  $k = k_0$ . So after a reflection about the slice  $t = h(k_0)$ , we obtain a complete rotational topological sphere.

ii) The proof for the case  $a < 0$  is analogous, taking into account that in this case  $\frac{2a}{a+b} < 0$  and  $(\cosh k)^{\frac{2a}{a+b}} - 1 < 0$ .

For the case  $\mathbb{S}^2 \times \mathbb{R}$ , assume  $a + b < 0$  and  $-b \leq c < 0$ . If  $\mathcal{S}$  were a rotational surface whose generating curve cuts orthogonally the rotation axis, there would exist a point  $p \in \mathcal{S}$  such that  $v^2(p) = 1$ . Our assumption on  $a, b$  and  $c$  implies  $a < -b \leq c$ , that is  $c - a > 0$ . Thus in such a point  $p \in \mathcal{S}$ , we would have  $K_e(p) = \frac{c - a}{a + b} < 0$ , a contradiction. This complete the proof.  $\square$

#### 4. VERTICAL AND HORIZONTAL HEIGHT ESTIMATES

In this section we consider a W-surface  $\Sigma$  isometrically immersed in  $M^2(\epsilon) \times \mathbb{R}$ , having positive extrinsic curvature. Once the extrinsic curvature is positive, the surface is orientable and we orient  $\Sigma$  in such way that the second fundamental form is positive definite. Let  $z$  be a conformal local parameter for the second fundamental form, in this parameter the first and second fundamental form of  $\Sigma$  are written as

$$(4.1) \quad I = Edz^2 + 2F|dz|^2 + \bar{E}d\bar{z}^2;$$

$$(4.2) \quad II = 2\rho|dz|^2,$$

where  $\rho$  is a positive function and  $\bar{z}$  denotes the conjugate of  $z$ . The extrinsic curvature of  $\Sigma$  is given by  $K_e = \frac{\rho^2}{D}$ , where  $D = F^2 - |E|^2 > 0$ , and we denote by  $K_i$  the intrinsic curvature of the surface.

**4.1. Some basic equations.** In this subsection we compute some equations which will be necessary to achieve the classification of W-surfaces.

**Lemma 4.1.** *Let  $\varphi : \Sigma \rightarrow M^2(\epsilon) \times \mathbb{R}$  be an isometric immersion in  $M^2(\epsilon) \times \mathbb{R}$ . Assume  $\Sigma$  is a W-surface having positive extrinsic curvature and that  $a + b$  is different from zero. Let  $N$  be the global unit normal vector field such that the second fundamental form of  $\Sigma$  is positive definite and  $z$  be a conformal parameter for the second fundamental form. Then, the following equations are satisfied:*

$$(4.3) \quad K_e = \frac{c - \epsilon av^2}{a + b}$$

$$(4.4) \quad \frac{\rho_{\bar{z}}}{\rho} = -\frac{\epsilon v \alpha}{\rho} - (\Gamma_{12}^1 - \Gamma_{22}^2) \quad (\text{Codazzi equation})$$

$$(4.5) \quad h_{z\bar{z}} = v\rho + \Gamma_{12}^1 h_z + \Gamma_{12}^2 h_{\bar{z}}$$

$$(4.6) \quad h_{zz} = \Gamma_{11}^1 h_z + \Gamma_{11}^2 h_{\bar{z}}$$

$$(4.7) \quad v_{\bar{z}} = -\frac{\alpha K_e}{\rho}$$

$$(4.8) \quad |T|^2 = 1 - v^2 = \frac{1}{D}(\alpha h_z + \bar{\alpha} h_{\bar{z}})$$

$$(4.9) \quad |h_z|^2 = -\frac{|\alpha|^2}{D} + F|T|^2,$$

where,  $\Gamma_{ij}^k$ ,  $i, j, k = 1, 2, 3$  are the Christoffel symbols associated to  $z$ ;  $E, F, \rho$  are terms of the first and second fundamental forms given by equations (4.1) and (4.2), and

$$(4.10) \quad \alpha := Fh_{\bar{z}} - \bar{E}h_z$$

$$(4.11) \quad D := F^2 - |E|^2$$

$$(4.12) \quad T = \frac{1}{D}(\alpha \partial_z + \bar{\alpha} \partial_{\bar{z}}).$$

*Proof.* This lemma is similar to [4, Lemma 3.1], for completeness we present a proof here. The idea is to write the compatibility equations in terms of the conformal parameter  $z$ . The compatibility equations for immersions in  $M^2(\epsilon) \times \mathbb{R}$  are described in [5].

Let  $\pi_2 : M^2(\epsilon) \times \mathbb{R} \longrightarrow \mathbb{R}$ ,  $\pi_2(p, t) = t$  be the projection on the second factor. We write  $\frac{\partial}{\partial t} = T + \nu N$ , where  $T$  is a tangent vector field to  $\Sigma$ . Once  $\frac{\partial}{\partial t}$  is the gradient in  $M^2(\epsilon) \times \mathbb{R}$  of the function  $\pi_2$ , the vector field  $T$ , tangent to  $\Sigma$ , is the gradient of the height function  $h := \pi_2|_{\Sigma}$ . Then,

$$T = \frac{1}{D}(\alpha \partial_z + \bar{\alpha} \partial_{\bar{z}}).$$

Observe that  $|T|^2 = 1 - \nu^2$ , and after a direct computation we obtain equation (4.8). On the other hand, by equation (4.10)  $h_z = \frac{1}{D}(E\alpha + F\bar{\alpha})$ , then

$$\begin{aligned} |h_z|^2 &= \frac{1}{D^2}(|\alpha|^2(|E|^2 + F^2) + F(E\alpha^2 + \bar{E}\bar{\alpha}^2)) \\ &= -\frac{|\alpha|^2}{D} + \frac{F}{D^2}(E\alpha^2 + 2F|\alpha|^2 + \bar{E}\bar{\alpha}^2) \\ &= -\frac{|\alpha|^2}{D} + F|T|^2, \end{aligned}$$

which proves equations (4.9).

Using the Gauss equation  $K_i = K_e + \epsilon\nu^2$ , the Weingarten equation  $aK_i + bK_e = c$  becomes

$$K_e = \frac{c - \epsilon a \nu^2}{a + b}.$$

The Codazzi equation is

$$(4.13) \quad \nabla_X \mathcal{A}Y - \nabla_Y \mathcal{A}X - \mathcal{A}[X, Y] = \epsilon\nu(\langle Y, T \rangle X - \langle X, T \rangle Y),$$

where  $\mathcal{A}$  is the shape operator of  $\Sigma$  and  $X, Y$  are tangent vector fields to  $\Sigma$ . For  $X = \partial_{\bar{z}}, Y = \partial_z$  the Codazzi equation is,

$$\nabla_{\partial_{\bar{z}}} \mathcal{A}\partial_z - \nabla_{\partial_z} \mathcal{A}\partial_{\bar{z}} = \epsilon\nu(h_z \partial_{\bar{z}} - h_{\bar{z}} \partial_z),$$

the scalar product of this equation with  $\partial_{\bar{z}}$  gives

$$\langle \nabla_{\partial_{\bar{z}}} \mathcal{A}\partial_z, \partial_{\bar{z}} \rangle - \langle \nabla_{\partial_z} \mathcal{A}\partial_{\bar{z}}, \partial_{\bar{z}} \rangle = \epsilon\nu(h_z \bar{E} - h_{\bar{z}} F) = -\epsilon\nu\alpha$$

$$\frac{\rho_{\bar{z}}}{\rho} + (\Gamma_{12}^1 - \Gamma_{22}^2) = -\frac{\epsilon\nu\alpha}{\rho},$$

which is the equation (4.4).

Taking the scalar product of the compatibility equation  $\nabla_X T = \nu \mathcal{A}X$  with  $\partial_{\bar{z}}$ , for  $X = \partial_z$  we get,

$$\langle \nabla_{\partial_z} T, \partial_{\bar{z}} \rangle = \nu \langle \mathcal{A} \partial_z, \partial_{\bar{z}} \rangle$$

$$h_{z\bar{z}} - \langle T, \nabla_{\partial_z} \partial_{\bar{z}} \rangle = \nu \rho,$$

then, we obtain equation (4.5)

$$h_{z\bar{z}} = \nu \rho + \Gamma_{12}^1 h_z + \Gamma_{12}^2 h_{\bar{z}}.$$

Similarly, taking the scalar product of the compatibility equation  $\nabla_X T = \nu \mathcal{A}X$  with  $\partial_z$ , for  $X = \partial_z$  we get equation (4.6).

From the compatibility equation  $d\nu(X) = -\langle \mathcal{A}X, T \rangle$ , for  $X = \partial_{\bar{z}}$ , we have

$$\nu_{\bar{z}} = -\langle \mathcal{A} \partial_{\bar{z}}, \frac{\alpha \partial_z + \bar{\alpha} \partial_{\bar{z}}}{D} \rangle = -\frac{\alpha K_e}{\rho}.$$

□

The equations on Lemma 4.1 enable us to rewrite  $h_{z\bar{z}}$  and  $\nu_{z\bar{z}}$ , in a more suitable form, it is done in the following Proposition.

**Proposition 4.2.** *Let  $\varphi : \Sigma \rightarrow M^2(\epsilon) \times \mathbb{R}$  be an isometric immersion in  $M^2(\epsilon) \times \mathbb{R}$ . Assume  $\Sigma$  is a W-surface having positive extrinsic curvature and that  $a + b$  is different from zero. Let  $N$  be the global unit normal vector field to  $\Sigma$  such that the second fundamental form is positive definite and  $z$  be a conformal parameter for the second fundamental form, then*

$$(4.14) \quad h_{z\bar{z}} = \frac{\nu \rho}{2K_e(a+b)} (2K_e(a+b) - \epsilon(2a+b)(1-\nu^2))$$

$$(4.15) \quad \nu_{z\bar{z}} = -\frac{\epsilon a \nu |\alpha|^2}{(a+b)D} - F \nu K_e,$$

where  $\alpha$  and  $D$  are defined in (4.10) and (4.11), respectively.

*Proof.* We start proving equation (4.14). Since  $\Sigma$  is a W-surface, taking the derivative of equation (4.3) with respect to  $\bar{z}$  and using equation (4.7), we obtain

$$(4.16) \quad \frac{(K_e)_{\bar{z}}}{2K_e} = \frac{\epsilon a \alpha \nu}{(a+b)\rho}.$$

On the other hand,  $K_e = \frac{\rho^2}{D}$ , then

$$(4.17) \quad \frac{(K_e)_{\bar{z}}}{2K_e} = \frac{\rho_{\bar{z}}}{\rho} - \frac{D_{\bar{z}}}{2D}.$$

As a consequence of (4.16) and (4.17), we have

$$(4.18) \quad \frac{\rho_{\bar{z}}}{\rho} - \frac{D_{\bar{z}}}{2D} = \frac{\epsilon a \alpha \nu}{(a+b)\rho}.$$

A direct computation, see [12, Lemma 8],

$$(4.19) \quad \Gamma_{12}^1 + \Gamma_{22}^2 = \frac{D_{\bar{z}}}{2D}.$$



The Codazzi equation (4.4) is equivalent to

$$\begin{aligned} \frac{\rho_{\bar{z}}}{\rho} - (\Gamma_{22}^2 + \Gamma_{12}^1) + 2\Gamma_{12}^1 &= -\frac{\epsilon \nu \alpha}{\rho} \\ \text{by (4.19),} \quad \frac{\rho_{\bar{z}}}{\rho} - \frac{D_{\bar{z}}}{2D} + 2\Gamma_{12}^1 &= -\frac{\epsilon \nu \alpha}{\rho} \\ \implies \\ \text{by (4.18),} \quad \Gamma_{12}^1 &= -\frac{\epsilon \alpha \nu (2a + b)}{2\rho (a + b)}. \end{aligned} \tag{4.20}$$

Since  $\Gamma_{12}^1 = \overline{\Gamma_{12}^2}$ , using equation (4.5), we have

$$\begin{aligned} h_{z\bar{z}} &= -\frac{\epsilon \nu (2a + b)}{2\rho (a + b)} (\alpha h_z + \bar{\alpha} h_{\bar{z}}) + \nu \rho \\ \text{by (4.8),} \quad h_{z\bar{z}} &= -\frac{\epsilon \nu (2a + b)}{2\rho (a + b)} (1 - \nu^2) D + \nu \rho \\ \implies \\ &= \nu \rho \left( 1 - \frac{\epsilon (2a + b)(1 - \nu^2)}{2K_e (a + b)} \right) \\ &= \frac{\nu \rho}{2K_e (a + b)} (2K_e (a + b) - \epsilon (2a + b)(1 - \nu^2)), \end{aligned}$$

which prove equation (4.14).

In order to prove equation (4.15), observe that by equations (4.7), (4.16) and (4.18), we have

$$\begin{aligned} \nu_{z\bar{z}} &= -\alpha_z \frac{K_e}{\rho} - \frac{2\epsilon a \nu |\alpha|^2 K_e}{\rho^2 (a + b)} + \frac{\alpha K_e}{\rho} \left( \frac{\epsilon a \bar{\alpha} \nu}{\rho (a + b)} + \frac{D_z}{2D} \right) \\ &= -\alpha_z \frac{K_e}{\rho} - \frac{\epsilon a \nu |\alpha|^2}{D (a + b)} + \frac{\alpha K_e}{\rho} \frac{D_z}{2D}. \end{aligned} \tag{4.21}$$

We claim that

$$\alpha_z = \alpha \frac{D_z}{D} + F \nu \rho. \tag{4.22}$$

Let us assume this equation for a moment. Then, a direct computation using equations (4.21) and (4.22) gives the equation (4.15), as desired. So, in order to finish the proof of the proposition, we need to prove that equation (4.22) holds. Recall  $\alpha = F h_{\bar{z}} - \bar{E} h_z$ . Then, using equations (4.5) and (4.6), we obtain

$$\begin{aligned} \alpha_z &= \langle \nabla_{\partial_z} \partial_z, \partial_{\bar{z}} \rangle h_{\bar{z}} + \langle \nabla_{\partial_z} \partial_{\bar{z}}, \partial_z \rangle h_{\bar{z}} - 2 \langle \nabla_{\partial_z} \partial_{\bar{z}}, \partial_{\bar{z}} \rangle h_z + F h_{z\bar{z}} - \bar{E} h_{zz} \\ &= \Gamma_{11}^1 (F h_{\bar{z}} - \bar{E} h_z) + \Gamma_{12}^1 (E h_{\bar{z}} - F h_z) + 2\Gamma_{12}^2 (F h_{\bar{z}} - \bar{E} h_z) + F \nu \rho, \\ &= \Gamma_{11}^1 \alpha - \Gamma_{12}^1 \bar{\alpha} + 2\Gamma_{12}^2 \alpha + F \nu \rho, \end{aligned}$$

a direct computation using equation (4.20) shows that  $\Gamma_{12}^2 \alpha - \Gamma_{12}^1 \bar{\alpha} = 0$ . Moreover, conjugating equation (4.19), we obtain

$$\alpha_z = \alpha(\Gamma_{11}^1 + \Gamma_{12}^2) + F \nu \rho = \alpha \left( \frac{D_z}{D} \right) + F \nu \rho,$$

as claimed. □

**4.2. A quadratic form on  $\Sigma$ .** In this section, we will define a quadratic form  $Q dz^2$  on  $\Sigma$  having the property that  $Q$  vanishes identically or its zeros are isolated with negative index.

Let  $\Sigma$  be a W-surface isometrically immersed in  $M^2(\epsilon) \times \mathbb{R}$  having positive extrinsic curvature, assume that  $a + b$  and  $2a + b$  are different from zero. For such a W-surface we introduce the quadratic forms

$$(4.23) \quad A := I + f(1 - \nu^2) dh^2,$$

$$(4.24) \quad Q dz^2 := (E + f(1 - \nu^2) h_z^2) dz^2,$$

where  $I$  is the first fundamental form of  $\Sigma$  given in (4.1) and  $f : [0, 1] \rightarrow \mathbb{R}$  is the real analytic function given by

$$(4.25) \quad f(t) = \frac{-\epsilon(2a + b)(c - \epsilon a)t - (c - \epsilon a)^2 + (c - \epsilon a(1 - t))^{\frac{2a+b}{a}}(c - \epsilon a)^{-\frac{b}{a}}}{\epsilon(a + b)(c - \epsilon a)t^2}.$$

*Remark 4.3.* We point out that

- (1) The quadratic form  $Q dz^2$  is the  $(2, 0)$ -part of  $A$ .
- (2) The Taylor series near zero of  $f$  is

$$f(t) = \sum_{n=0}^{n=\infty} a_n t^n,$$

where  $a_n = \frac{\epsilon^{n+1}}{(a + b)(c - \epsilon a)^{(1+n)}(n + 2)!} \prod_{j=0}^{n+1} (2a + b - j a)$ . The convergence radius of this series is  $\frac{|c - \epsilon a|}{|a|} > 0$ . So,  $f$  is real analytic on  $[0, 1]$ .

The extrinsic curvature of the pair  $(II, A)$  is, see [12]

$$(4.26) \quad \begin{aligned} K(II, A) &= \frac{(F + f(1 - \nu^2) |h_z|^2)^2 - |E + f(1 - \nu^2) h_z^2|^2}{\rho^2} \\ &= \frac{F^2 - |E|^2}{\rho^2} - \frac{f(1 - \nu^2) (\bar{E} h_z^2 + -2F |h_z|^2 + E h_z^2)}{\rho^2} \\ &\stackrel{\text{by (4.10),}}{\implies} \frac{1}{K_e} + \frac{f(1 - \nu^2) D(\alpha h_z + \bar{\alpha} h_{\bar{z}})}{\rho^2} \\ &\stackrel{\text{by (4.8),}}{\implies} \frac{1}{K_e} (1 + f(1 - \nu^2) |T|^2), \end{aligned}$$

in particular, once  $|Q|^2 = |E + f(1 - \nu^2) h_z^2|^2$ , using the first and fourth lines of (4.26), we have

$$(4.27) \quad |Q|^2 = (F + f(1 - \nu^2) |h_z|^2)^2 - D(1 + f(1 - \nu^2) |T|^2).$$

The next result is the key lemma, which gives a estimate of  $|Q_{\bar{z}}|$  in terms of the function  $|Q|$ , more precisely

**Lemma 4.4.** *Let  $\varphi : \Sigma \longrightarrow M^2(\epsilon) \times \mathbb{R}$  be an isometric immersion in  $M^2(\epsilon) \times \mathbb{R}$ . We assume  $\Sigma$  is a  $W$ -surface having positive extrinsic curvature. We also suppose that  $a + b$  and  $2a + b$  are different from zero. Let  $z$  be a conformal parameter for the second fundamental form. Then*

$$(4.28) \quad |Q_{\bar{z}}| \leq \frac{2|\nu \rho h_z^3 f'(1 - \nu^2)|}{D} |Q|,$$

where  $Q$  and  $D$  are defined in (4.24) and (4.11), respectively, and  $f'(t)$  is the derivative of  $f$  at  $t$ .

*Proof.* The derivative of the function  $Q$  with respect to  $\bar{z}$  is

$$(4.29) \quad Q_{\bar{z}} = E_{\bar{z}} + 2f(1 - \nu^2) h_z h_{z\bar{z}} - 2\nu \nu_{\bar{z}} f'(1 - \nu^2) h_z^2.$$

Let us determine the expression of  $E_{\bar{z}}$ . Observe that the Christoffel symbols with respect to the conformal parameter  $z$  satisfies,  $\Gamma_{12}^1 = \overline{\Gamma_{12}^2}$ . Using equations (4.20) and (4.10), we have

$$\begin{aligned} E_{\bar{z}} &= \partial_{\bar{z}} \langle \partial_z, \partial_z \rangle = 2(\Gamma_{12}^1 E + \Gamma_{12}^2 F) \\ &= -\frac{\epsilon \nu (2a + b)}{\rho(a + b)} (\alpha E + \bar{\alpha} F) \\ &= -\frac{\epsilon \nu (2a + b)}{\rho(a + b)} D h_z, \end{aligned}$$

then, since  $K_e = \frac{\rho^2}{D}$ , we obtain

$$(4.30) \quad E_{\bar{z}} = -\frac{\epsilon \nu (2a + b) \rho h_z}{K_e(a + b)}.$$

By equations (4.14), (4.7), (4.29) and (4.30), we have

$$Q_{\bar{z}} = \nu \rho h_z \left( -\frac{\epsilon(2a + b)}{K_e(a + b)} + f(1 - \nu^2) \left( \frac{2K_e(a + b) - \epsilon(2a + b)(1 - \nu^2)}{K_e(a + b)} \right) + f'(1 - \nu^2) \frac{2\alpha K_e h_z}{\rho^2} \right),$$

a direct computation shows that for  $2a + b \neq 0$ ,

$$-\frac{\epsilon(2a + b)}{K_e(a + b)} + f(1 - \nu^2) \left( \frac{2K_e(a + b) - \epsilon(2a + b)(1 - \nu^2)}{K_e(a + b)} \right) = -(1 - \nu^2) f'(1 - \nu^2).$$

Using the above equation, we obtain

$$\begin{aligned}
Q_{\bar{z}} &= \nu \rho h_z f'(1 - \nu^2) \left( -(1 - \nu^2) + \frac{2\alpha h_z}{D} \right) \\
&\stackrel{\text{by (4.8),}}{\implies} = \nu \rho h_z f'(1 - \nu^2) \left( \frac{\alpha h_z - \bar{\alpha} h_{\bar{z}}}{D} \right) \\
&\stackrel{\text{by (4.10),}}{\implies} = \frac{\nu \rho h_z f'(1 - \nu^2)}{D} (E h_z^2 - \bar{E} h_{\bar{z}}^2) \\
&\stackrel{\text{by (4.24),}}{\implies} = \frac{\nu \rho h_z f'(1 - \nu^2)}{D} ((Q - f(1 - \nu^2) h_z^2) h_{\bar{z}}^2 - (\bar{Q} - f(1 - \nu^2) h_{\bar{z}}^2) h_z^2) \\
&= \frac{\nu \rho h_z f'(1 - \nu^2)}{D} (Q h_{\bar{z}}^2 - \bar{Q} h_z^2),
\end{aligned}$$

then

$$|Q_{\bar{z}}| \leq \frac{2|\nu \rho h_z^3 f'(1 - \nu^2)|}{D} |Q|,$$

as desired. □

Lemma 4.4 is used to apply [11, Lemma 2.7.1] and obtain an important property of the function  $Q$ .

**Proposition 4.5.** *Let  $\varphi : \Sigma \longrightarrow M^2(\epsilon) \times \mathbb{R}$  be an isometric immersion in  $M^2(\epsilon) \times \mathbb{R}$ . Assume  $\Sigma$  is a  $W$ -surface having positive extrinsic curvature. Moreover, we suppose  $a + b$  and  $2a + b$  are different from zero. Consider  $\Sigma$  as a Riemann surface with the conformal structure induced by its second fundamental form. Then, the quadratic form  $Q dz^2$ , where  $Q : \Sigma \longrightarrow \mathbb{C}$  is defined in (4.24), vanishes identically or its zeros are isolated with negative index.*

A direct consequence of Proposition 4.5 is

**Proposition 4.6.** *Let  $\varphi : \Sigma \longrightarrow M^2(\epsilon) \times \mathbb{R}$  be an isometric immersion in  $M^2(\epsilon) \times \mathbb{R}$ . Assume  $\Sigma$  is a  $W$ -surface having positive extrinsic curvature. Moreover, we suppose  $a + b$  and  $2a + b$  are different from zero. Consider  $\Sigma$  as a Riemann surface with the conformal structure induced by its second fundamental form. If  $\Sigma$  is a topological sphere, then the function  $Q$  is identically null on  $\Sigma$ .*

**4.3. Vertical height estimates.** This section is devoted to give a vertical height estimates for some  $W$ -surfaces, more precisely

**Theorem 4.7** (Vertical height estimates). *Let  $\varphi : \Sigma \longrightarrow \mathbb{H}^2 \times \mathbb{R}$  be a compact graph on a domain  $\Omega \subset \mathbb{H}^2$  whose boundary is contained in the slice  $\mathbb{H}^2 \times \{0\}$ . Assume  $\Sigma$  is a  $W$ -surface having positive extrinsic curvature. Moreover, suppose  $2a + b$  is different from zero,  $a + b > 0$  and  $c > 0$ . Then there exists a constant  $C_0$  which depends only on  $K_e, a, b, c$ , such that the height function  $h$ , satisfies  $h(p) \leq C_0$  for all  $p$  in  $\Sigma$ .*

*Proof.* The idea of this proof is to cook up a sub-harmonic function  $\phi = h + g(v)$  on  $\Sigma$  having non-positive boundary values, where  $g : [-1, 0] \longrightarrow \mathbb{R}$  is to be determined. In order to compute  $\phi_{z\bar{z}}$ , we

calculated  $(g(\nu))_{z\bar{z}}$ . Taking into account equations (4.3), (4.15), (4.7) and (4.9), we obtain

$$\begin{aligned}
(g(\nu))_{z\bar{z}} &= (\nu_z g'(\nu))_{\bar{z}} = \nu_{z\bar{z}} g'(\nu) + |\nu_z|^2 g''(\nu) \\
&= \frac{|\alpha|^2}{D} \left( \frac{av}{(a+b)} g'(\nu) + K_e g''(\nu) \right) - F \nu K_e g'(\nu) \\
&= F \left( \left( \frac{av(1-\nu^2)}{(a+b)} - \nu K_e \right) g'(\nu) + K_e(1-\nu^2) g''(\nu) \right) + |h_z|^2 \left( -\frac{av}{(a+b)} g'(\nu) - K_e g''(\nu) \right) \\
&= K_e \left( F \left( \left( \frac{av(1-\nu^2)}{(c+av^2)} - \nu \right) g'(\nu) + (1-\nu^2) g''(\nu) \right) + |h_z|^2 \left( -\frac{av}{(c+av^2)} g'(\nu) - g''(\nu) \right) \right).
\end{aligned}$$

Let  $g : [-1, 0] \rightarrow \mathbb{R}$  be a real function whose derivative is given by

$$g'(t) = M \sqrt{\frac{\left(\frac{c+a}{c+at^2}\right)^{\frac{a+b}{a}} - 1}{(1-t^2)(c+at^2)}},$$

where  $M$  is a constant depending only on  $a, b, c$  defined by

$$(4.31) \quad M = \begin{cases} \frac{\max_{\nu \in [-1, 0]} \left( 1 + \frac{(2a+b)(1-\nu^2)}{2(c+av^2)} \right)}{\min_{\nu \in [-1, 0]} \left( \sqrt{\frac{1}{c+av^2}} \left( \frac{c+a}{c+av^2} \right)^{\frac{a+b}{a}} \right)}, & \text{if } \max_{\nu \in [-1, 0]} \left( 1 + \frac{(2a+b)(1-\nu^2)}{2(c+av^2)} \right) > 0; \\ 1 & \text{if } \max_{\nu \in [-1, 0]} \left( 1 + \frac{(2a+b)(1-\nu^2)}{2(c+av^2)} \right) \leq 0, \end{cases}$$

here  $\max_{s \in [s_0, s_1]} (u(s))$  and  $\min_{s \in [s_0, s_1]} (u(s))$  are the maximum and minimum of the function  $u(s)$  for  $s$  in  $[s_0, s_1]$ .

A direct computation shows that

$$(4.32) \quad \left( \frac{av(1-\nu^2)}{(c+av^2)} - \nu \right) g'(\nu) + (1-\nu^2) g''(\nu) = -\frac{\nu g'(\nu)}{1 + (1-\nu^2) f(1-\nu^2)},$$

where the real function  $f(t)$  is defined in (4.25). Also, we have

$$(4.33) \quad -\frac{av}{(c+av^2)} g'(\nu) - g''(\nu) = f(1-\nu^2) \left( -\frac{\nu g'(\nu)}{1 + (1-\nu^2) f(1-\nu^2)} \right).$$

Then, by (4.32) and (4.33),

$$(4.34) \quad g_{z\bar{z}} = -\frac{\nu K_e g'(\nu)}{1 + (1-\nu^2) f(1-\nu^2)} (F + |h_z|^2 f(1-\nu^2)).$$

Observe that  $(F + |h_z|^2 f(1-\nu^2))$  is positive on  $\Sigma$ , in fact, since the extrinsic curvature  $K(II, A)$  on equation (4.26) of the pair  $(II, A)$  is positive, the quadratic form  $A$  is positive definite or negative definite which implies that either  $(F + |h_z|^2 f(1-\nu^2))$  is positive in  $\Sigma$  or it is negative everywhere. At the highest point  $h_z = 0$  and we have  $(F + |h_z|^2 f(1-\nu^2)) = F$  is positive, so we conclude that  $(F + |h_z|^2 f(1-\nu^2))$  is positive in  $\Sigma$ .

In order to compute  $\phi_{z\bar{z}}$  it is worth to write  $g'(t)$  as

$$g'(t) = M \sqrt{\frac{1 + (1 - t^2) f(1 - t^2)}{K_e((c + a t^2) - (a + b)(1 - t^2)(1 + (1 - t^2) f(1 - t^2)))}},$$

keeping this in mind, using equations (4.14), (4.34) and (4.27), we obtain

$$\begin{aligned} (\phi)_{z\bar{z}} &= \nu \left( \rho + \frac{\rho(2a + b)(1 - \nu^2)}{2K_e(a + b)} - \frac{K_e g'(\nu)}{1 + (1 - \nu^2) f(1 - \nu^2)} (F + |h_z|^2 f(1 - \nu^2)) \right) \\ &= \nu \left( \rho + \frac{\rho(2a + b)(1 - \nu^2)}{2K_e(a + b)} - \frac{K_e g'(\nu)}{1 + (1 - \nu^2) f(1 - \nu^2)} \sqrt{|Q|^2 + \frac{\rho^2(1 + (1 - \nu^2) f(1 - \nu^2))}{K_e}} \right) \\ &\geq \nu \rho \left( 1 + \frac{(2a + b)(1 - \nu^2)}{2K_e(a + b)} - \frac{K_e g'(\nu)}{1 + (1 - \nu^2) f(1 - \nu^2)} \sqrt{\frac{1 + (1 - \nu^2) f(1 - \nu^2)}{K_e}} \right) \\ &= \nu \rho \left( 1 + \frac{(2a + b)(1 - \nu^2)}{2K_e(a + b)} - M \sqrt{\frac{1}{(c + a \nu^2) - (a + b)(1 - \nu^2)(1 + (1 - \nu^2) f(1 - \nu^2))}} \right) \\ &= \nu \rho \left( 1 + \frac{(2a + b)(1 - \nu^2)}{2(c - \epsilon a \nu^2)} - M \sqrt{\frac{1}{c + a \nu^2} \left( \frac{c + a}{c + a \nu^2} \right)^{\frac{a+b}{a}}} \right), \end{aligned}$$

so, the definition of  $M$  implies that  $\phi_{z\bar{z}} \geq 0$ . Taking

$$g(\nu) = \int_0^\nu g'(t) dt,$$

we have  $\Delta^{II}(h + g(\nu)) = \frac{2}{\rho}(h + g(\nu))_{z\bar{z}} \geq 0$  in  $\Sigma$ , where  $\Delta^{II}$  is the laplacian with respect to the second fundamental form. Moreover,  $h + g(\nu)$  is non-positive on the boundary of  $\Sigma$ , once  $g'(\nu)$  is non-negative and  $\nu \leq 0$ , then we have that  $h + g(\nu)$  is non-positive everywhere. In particular, the maximum of the function  $h$  is

$$C_0 = \int_{-1}^0 g'(t) dt.$$

□

**4.4. Horizontal heigh estimates.** In this section we will see that the horizontal height for a class of compact embedded W-surfaces in  $\mathbb{H}^2 \times \mathbb{R}$  with boundary on a vertical plane is bounded.

Let  $\varphi : \Sigma \rightarrow M^2(\epsilon) \times \mathbb{R}$  be an isometric immersion from a oriented surface  $\Sigma$ . Recall  $\Sigma$  is a W-surface if the Weingarten function

$$K_e - \frac{c - a\epsilon\nu^2}{a + b}$$

vanishes identically. Observe that we may regard  $K_e - \frac{c - a\epsilon\nu^2}{a + b} = 0$  as a second order partial differential equation. From this point of view, once  $\nu$  depends only on the first derivative of the immersion, it can be showed that the partial differential equation  $K_e - \frac{c - a\epsilon\nu^2}{a + b} = 0$  is absolutely

elliptic if  $K_e > 0$ . So, if  $\Sigma$  is a W-surface having positive extrinsic curvature the interior and the boundary maximum principle, in the sense of Hopf, hold.

Let  $\varphi_j : \Sigma_j \rightarrow M^2(\epsilon) \times \mathbb{R}$ ,  $j = 1, 2$ , be two isometric immersions. Assume  $\Sigma_j$  is a W-surface having positive extrinsic curvature. Let  $N_j$  be the global unit normal vector field to  $\Sigma_j$  such that the second fundamental form is positive definite. Let  $p \in \Sigma_1 \cap \Sigma_2$  and assume that  $N_1(p) = N_2(p)$ . Once  $N_1(p) = N_2(p)$ , there is a neighbourhood  $U_j \subset \Sigma_j$  of  $p$  such that  $U_j$  is a graph in exponential coordinates of a function  $f_j$  defined on a neighbourhood  $\mathcal{D}$  of the origin of  $T_p\Sigma_1 = T_p\Sigma_2$  ( $T_p\Sigma_j$  is the tangent plane of  $\Sigma_j$  at  $p$ ). Since the extrinsic curvature of  $\Sigma_j$  is positive,  $f_j$  is a positive function (for  $\mathcal{D}$  small enough). We say that  $\Sigma_1$  is above  $\Sigma_2$ , which we denote by  $\Sigma_1 \geq \Sigma_2$ , in a neighborhood of  $p$  if  $f_1 \geq f_2$  in  $\mathcal{D}$ .

Under this notation, we have the following important theorem.

**Theorem 4.8** (Hopf Maximum Principle, [10]). *Let  $\varphi_j : \Sigma_j \rightarrow M^2(\epsilon) \times \mathbb{R}$ ,  $j = 1, 2$ , be two isometric immersions. Assume  $\Sigma_j$  is a W-surface having positive extrinsic curvature. Let  $N_j$  be the global unit normal vector field to  $\Sigma_j$  such that the second fundamental form associated to  $\Sigma_j$  is positive definite.*

*Suppose that,*

- i)  $\Sigma_1$  and  $\Sigma_2$  are tangent at an interior point  $p \in \Sigma_1 \cap \Sigma_2$ , or
- ii) there exists  $p \in \partial\Sigma_1 \cap \partial\Sigma_2$  such that both  $T_p\Sigma_1 = T_p\Sigma_2$  and  $T_p\partial\Sigma_1 = T_p\partial\Sigma_2$ ,

*furthermore, suppose that the unit normal vector fields of  $\Sigma_1$  and  $\Sigma_2$  coincide at  $p$ . If  $\Sigma_1 \geq \Sigma_2$  in a neighbourhood  $U_j \subset \Sigma_j$  of  $p$ , then  $\Sigma_1 = \Sigma_2$  in  $U_1 = U_2$ .*

In order to state the horizontal height estimate, recall a vertical plane in  $\mathbb{H}^2 \times \mathbb{R}$  is the product  $\gamma \times \mathbb{R}$  of a complete geodesic  $\gamma \subset \mathbb{H}^2$  with the real line  $\mathbb{R}$ .

**Theorem 4.9** (Horizontal height estimates). *Let  $\varphi : \Sigma \rightarrow \mathbb{H}^2 \times \mathbb{R}$  be an isometric immersion. Suppose  $\Sigma$  is a compact embedded W-surface having positive extrinsic curvature whose boundary is contained in a vertical plane  $P$ . Moreover, assume  $a + b > 0$  and  $c > 0$ . Then the distance from  $\Sigma$  to  $P$  is bounded; i.e., there exists a constant  $c_0$  depending on  $a, b, c$ , independent of  $\Sigma$ , such that*

$$\text{dist}(q, P) \leq c_0, \text{ for all } q \in \Sigma.$$

Once the interior and boundary maximum principle hold for W-surfaces  $\Sigma$  isometrically immersed in  $M^2(\epsilon) \times \mathbb{R}$  having positive extrinsic curvature, the proof of [8, Theorem 6.2] applies to our setting with the exception that the proof use the maximum principle to compare  $\Sigma$  to a surface  $\Sigma_0$  that in our case is the rotational topological sphere presented in Section 3.1.

## 5. PROPERLY EMBEDDED W-SURFACES WITH FINITE TOPOLOGY AND ONE TOP END

We begin this section with the following definition.

**Definition 5.1.** Let  $\varphi : \Sigma \rightarrow M^2(\epsilon) \times \mathbb{R}$  be an isometric immersion from a complete surface. Then:

- (1) We say that  $\Sigma$  has a top end  $\mathcal{E}$  (respectively, a bottom end) if for any divergent sequence  $\{q_j\} \subset \mathcal{E}$ , the height function goes to  $+\infty$  (respectively,  $-\infty$ ).
- (2) For the case  $\mathbb{H}^2 \times \mathbb{R}$ , we say that  $\Sigma$  has a simple end if the boundary at infinity of the projection on the first factor  $\pi_1(\Sigma) \subset \mathbb{H}^2 \times \{0\}$  is a unique point  $\theta_0$  and in addition, for each vertical plane  $P$  whose boundary at infinity does not contain  $\theta_0$ , the intersection of  $P$  and  $\Sigma$  is either empty or a compact set. Here we are denoting by  $\pi_1 : M^2(\epsilon) \times \mathbb{R} \rightarrow M^2(\epsilon)$ ,  $\pi_1(p, t) = p$ , the projection on the first factor; and as usual, we identify the base space  $M^2(\epsilon)$  with its horizontal lift  $M^2(\epsilon) \times \{0\}$ .

Recall that, there is no properly embedded complete surface in  $\mathbb{H}^2 \times \mathbb{R}$  having positive constant extrinsic curvature with finite topology and one top (or bottom) end, see [8, Theorem 7.2]. In this section we extend this result to some  $W$ -surfaces.

For fixed real numbers  $a, b$  and  $c$  such that  $a + b > 0$  and  $c > 0$ , we denote by  $\mathcal{S}_c(a, b)$  the rotational topological sphere in  $\mathbb{H}^2 \times \mathbb{R}$  whose intrinsic and positive extrinsic curvatures satisfy the equation

$$(5.1) \quad aK_i + bK_e = c,$$

such rotational topological sphere was given in Section 3.1. We denote by  $c_1 = 2\kappa_0$  the horizontal diameter of  $\mathcal{S}_c(a, b)$ , where  $\kappa_0$  satisfies

$$(\cosh \kappa_0)^{\frac{a}{a+b}} = \sqrt{\frac{a+c}{c}}.$$

The following lemma extend the Plane Separation Lemma given in [14, Lemma 2.4] to properly embedded  $W$ -surface having positive extrinsic curvature. Using the Maximum Principle (Theorem 4.8), the proof of Lemma 5.2 is similar to the one of [14, Lemma 2.4], so we will not present a proof here.

**Lemma 5.2** (Plane Separation Lemma). *Let  $\varphi : \Sigma \rightarrow \mathbb{H}^2 \times \mathbb{R}$  be an isometric properly embedded  $W$ -surface having positive extrinsic curvature. Assume  $\Sigma$  has finite topology and a top (or bottom) end. Moreover, suppose  $a + b > 0$  and  $c > 0$ . Let  $P_1^+$  and  $P_2^+$  be two disjoint half-spaces determined by vertical planes  $P_1$  and  $P_2$ , respectively. If the distance between  $P_1$  and  $P_2$  is larger than the horizontal diameter  $c_1$  of the rotational topological sphere  $\mathcal{S}_c(a, b)$ . Then, either  $\Sigma \cap P_1^+$  or  $\Sigma \cap P_2^+$  consist entirely of compact components.*

As a consequence from Plane Separation Lemma and horizontal height estimates, we have the following theorem.

**Theorem 5.3.** *Let  $\varphi : \Sigma \rightarrow \mathbb{H}^2 \times \mathbb{R}$  be an isometric immersion. Assume  $\Sigma$  is a complete  $W$ -surface having positive extrinsic curvature and finite topology with a top (or a bottom) end. Moreover, suppose  $a + b > 0$  and  $c > 0$ . Then  $\Sigma$  is contained in a vertical cylinder  $\alpha \times \mathbb{R}$  in  $\mathbb{H}^2 \times \mathbb{R}$ , where  $\alpha \subset \mathbb{H}^2 \times \{0\}$  is a circle.*

*Proof.* First, observe that, since  $\Sigma$  has positive extrinsic curvature then  $\Sigma$  is properly embedded, see [8, Theorem 3.1]. We take the disk model for the hyperbolic plane  $\mathbb{H}^2$ . Up to an isometry of the ambient space, we can assume that the point  $\mathcal{O} = (0, 0)$  belongs to  $\Sigma$ , here  $\mathbf{0}$  denotes the origin of  $\mathbb{H}^2$ . Let  $\gamma : [0, +\infty) \rightarrow \mathbb{H}^2 \times \mathbb{R}$  be any horizontal geodesic starting at  $\mathcal{O}$  parameterized by arc length. We denote by  $P(s)$ ,  $s \in [0, +\infty)$ , the vertical plane passing through  $\gamma(s)$  orthogonal to  $\gamma$ .

**Claim.**— There exists a constant  $c_2$ , independent of  $\gamma$ , such that, if  $s_0 > c_2$ , then the half-space determined by  $P(s_0)$  that does not contain the point  $\mathcal{O}$  is disjoint from  $\Sigma$ .

**Proof of the Claim.**—We choose  $R > \max\{c_0, c_1\}$ , where  $c_0$  and  $c_1$  are the constant given by Theorem 4.9 and the Plane Separation Lemma, respectively. Denote by  $P^+(R)$  the half-space determined by  $P(R)$  containing the point  $\mathcal{O}$  and by  $P^-(2R)$  the half-space determined by  $P(2R)$  which does not contain the point  $\mathcal{O}$ . By the Plane Separation Lemma applied to  $\Sigma$ , we have

- a)  $\Sigma \cap P^+(R)$  has only compact components, or
- b)  $\Sigma \cap P^-(2R)$  has only compact components.

By Theorem 4.9, if a) were true, the distance between the plane  $P(R)$  and the point  $\mathcal{O} \in \Sigma \cap P^+(R)$  would be at most  $c_0$ . Once this horizontal distance is  $R > c_0$ , a) cannot occur. So b) holds. Again, by Theorem 4.9, the maximum distance between  $\Sigma \cap P^-(2R)$  and the plane  $P(2R)$  is at most  $c_0$ ; hence  $\Sigma$  is disjoint from the half-space determined by  $P(2R + c_0)$  which does not contain the point  $\mathcal{O}$ . Choosing the constant  $c_2 = 2 \max\{c_0, c_1\} + c_0$ , the Claim is proved.



The claim guarantees that  $\Sigma$  is contained in the vertical cylinder  $\alpha \times \mathbb{R}$ , here  $\alpha$  is a circle centered at the origin of  $\mathbb{H}^2 \times \{0\}$  having radius  $c_2$ .  $\square$

We finalize this section with a non-existence theorem.

**Theorem 5.4.** *There is no complete W-surface having positive extrinsic curvature, with  $a + b > 0$ ,  $2a + b \neq 0$  and  $c > 0$ , having finite topology and one top (or bottom) end in the product space  $\mathbb{H}^2 \times \mathbb{R}$ .*

*Proof.* Assume by contradiction that there exists such a W-surface  $\Sigma$  satisfying the hypothesis. Let  $E(t) = \mathbb{H}^2 \times \{t\}$  denote the horizontal slice at height  $t$ . By Theorem 5.3,  $\Sigma$  is contained in a vertical cylinder and since it has only one top (or bottom) end,  $\Sigma$  is bounded from either above or below. Up to an isometry of the ambient space, we can assume that  $\Sigma$  is bounded from below and a lowest point of  $\Sigma$  lie in the slice  $E(0)$ . As reflections with respect to the slices  $E(t)$  are isometries of  $\mathbb{H}^2 \times \mathbb{R}$ , we perform Alexandrov reflection to  $\Sigma$  with respect to the slice  $E(t)$  for  $t > 0$ . Since  $\Sigma$  is contained in a vertical cylinder, the maximum principle ensures that no accident can occur moving  $E(t)$  up, that is, for all  $t \geq 0$ , there is no point  $p \in \Sigma \cap E(t)$  such that  $\Sigma$  is orthogonal to  $E(t)$ . Moreover, denoting by  $\Sigma_t^*$  the reflection of  $\Sigma \cap (\mathbb{H}^2 \times [0, t])$  with respect  $E(t)$ , by the maximum principle, for all  $t > 0$ , there is no contact point between  $\Sigma_t^*$  and  $\Sigma \cap (\mathbb{H}^2 \times (t, +\infty))$ . Hence, for any  $t > 0$ , the part of  $\Sigma$  below  $E(t)$  is a vertical graph. But one can choose  $t$  larger enough, so that we have a contradiction with Theorem 4.7.  $\square$

## 6. THE MAIN THEOREM.

This last section is devoted to prove the main theorem.

**Theorem 6.1.** *Let  $\varphi : \Sigma \rightarrow M^2(\epsilon) \times \mathbb{R}$  be an isometric immersion. Assume  $\Sigma$  is a complete W-surface having positive extrinsic curvature. We suppose that  $2a + b$  is different from zero, then  $\Sigma$  is a topological rotational sphere described on Section 3.1 if*

- (A1) either  $a + b > 0$  and  $c > 0$ ,
- (A2) or for  $a + b < 0$ ,
  - (i)  $\epsilon = 1$  and  $c < -b$ ;
  - (ii)  $\epsilon = -1$  and  $c < 0$ .

*Proof.* From [8, Theorem 3.1] and [9, Theorem 2.4], once  $\Sigma$  has positive extrinsic curvature,  $\Sigma$  is either homeomorphic to  $\mathbb{R}^2$  or homeomorphic to  $S^2$ . So, using Lemma 2.3, Theorem 4.9 and Theorem 5.4 if (A1) or (A2) is satisfied then  $\Sigma$  is a topological sphere. As a consequence, Proposition 4.6 says that the quadratic differential form  $Q$  vanishes identically over  $\Sigma$ .

Let  $(u, v)$  be a local doubly orthogonal coordinates for the first and second fundamental form, in these coordinates

$$I = \mathbf{E} du^2 + \mathbf{G} dv^2$$

$$II = \kappa_1 \mathbf{E} du^2 + \kappa_2 \mathbf{G} dv^2,$$

where  $\kappa_1, \kappa_2$  are the principal curvatures of  $\Sigma$ . These coordinates are available on the interior of the set of umbilical points and also on a neighborhood of non umbilical points. So, the set of points where the coordinates  $(u, v)$  are available is dense on  $\Sigma$ , thus, properties obtained on this set are extended to  $\Sigma$  by continuity.

Since  $Q$  vanishes on  $\Sigma$ , the quadratic form  $A$ , defined on (4.23) is conformal to the second fundamental form  $II$ . It implies that  $h_u h_v = 0$ . Without lost of generality, we may assume that  $h_u = 0$  in

the neighborhood where  $(u, v)$  is available. Then, since  $A$  and  $II$  are conformal and  $h_u = 0$ ,

$$(6.1) \quad A = \mathbf{E} du^2 + (\mathbf{G} + h_v^2) dv^2 = \frac{1}{\kappa_1} II.$$

First, we prove that  $\Sigma$  is invariant under a one parameter group of isometries. Second, we show that an orbit of this one parameter group of isometry is a circle.

Proceeding as in Lemma 4.1, we write some compatibility equations with respect to the coordinates  $(u, v)$ , and we obtain

$$(6.2) \quad \nu_u = 0$$

$$(6.3) \quad \frac{\mathbf{E}_v}{2\mathbf{E}}(\kappa_2 - \kappa_1) = \epsilon \nu h_v + (\kappa_1)_v$$

$$(6.4) \quad \frac{\mathbf{G}_u}{2\mathbf{G}}(\kappa_2 - \kappa_1) = -(\kappa_2)_u$$

$$(6.5) \quad h_{uv} = 0 = \frac{\mathbf{G}_u}{2\mathbf{G}} h_v.$$

From equation (6.2) we obtain that  $\nu$  does not depend on  $u$ . As the extrinsic curvature of  $\Sigma$  is positive, no open neighbourhood of  $\Sigma$  is contained in a slice, therefore,  $h_v$  does not vanish in any open set where  $(u, v)$  are available, then equation (6.5) implies  $G_u = 0$ . Thus, by the Codazzi equation (6.4),  $(\kappa_2)_u = 0$ . Since the extrinsic curvature is  $K_e = \frac{c - \epsilon a v^2}{a + b}$  and  $\nu$  does not depend on  $u$  neither does  $K_e$ . On the other hand,  $K_e = \kappa_1 \kappa_2$  which implies that  $(\kappa_1)_u = 0$ .

The variables  $(u, v)$  are available in the interior set of umbilical points and on a neighborhood of non umbilical points. Let us assume for a moment that we are working on a neighborhood free of umbilical points. Then, by the Codazzi equation 6.3, we may write  $\mathbf{E} = \mathbf{E}_1(u)\mathbf{E}_2(v)$ . Considering the new variables

$$x := \sqrt{\mathbf{E}_1(u)} du \quad y := v,$$

we conclude that the first and second fundamental forms of  $\Sigma$ ,  $h$  and  $\nu$  depend only on  $y$ . Then,  $\varphi(x, y)$  and  $\varphi(x + x_0, y)$  only differ by an isometry of the ambient space, in other words, the immersion is invariant under one parameter group of isometries of the ambient space, given by the transformation  $(x, y) \mapsto (x + t, y)$ , see [5]. Once we know that  $\Sigma$  is a topological sphere, we conclude that  $\Sigma$  is invariant by the group of rotations of  $M^2(\epsilon)$ .

It remains to analyse the case where the coordinates  $(u, v)$  are defined on a neighbourhood in the interior of umbilical points. In this case,

$$I = \mathbf{E} du^2 + \mathbf{G} dv^2$$

$$II = \kappa_1(\mathbf{E} du^2 + \mathbf{G} dv^2)$$

$$A = \mathbf{E} du^2 + (\mathbf{G} + f(1 - v^2) h_v^2) dv^2 = \frac{1}{\kappa_1} II.$$

In particular,  $\mathbf{G} + f(1 - v^2) h_v^2 = \mathbf{G}$  which implies that  $h_v$  vanishes identically in this neighborhood. Then, once we are working on a neighborhood of umbilical points we conclude that the height function is constant, which implies that  $\Sigma$  is contained in a slice. This gives a contradiction, since the extrinsic curvature of the surface is positive. Then, there is no such a neighborhood of umbilical points.  $\square$

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